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Approximate Fixed Point Results for $(\alpha-\eta)-$Type and $(\beta-\psi)-$Type Fuzzy Contractive Mappings in $b-$Fuzzy Metric Spaces

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Abstract

This article contain the concepts of fuzzy $(\alpha-\eta)$ and fuzzy $(\beta-\psi)-$generalized proximal contractive mappings in the setup of $b-$fuzzy metric spaces. We proved the existence of coincidence and best proximity points of fuzzy $(\alpha-\eta)-$ and fuzzy $(\beta-\psi)-$generalized proximal contractions on $b-$fuzzy metric spaces. Some nontrivial examples are provided to elaborate the fact that the obtained results are potential generalizations of comparable existing results. Our results unify and complement various known results in the existing literature.

Keywords: $b-$fuzzy metric space; fuzzy $(\alpha-\eta)-$generalized proximal contraction; $(\beta-\psi)-$generalized proximal contraction; optimal coincidence point; $\alpha-$ admissible mapping; α_R- admissible mapping; $\beta-$ admissible mapping; β_R- admissible mapping.

1 Introduction and Preliminaries

The theory of fixed point has an important role in approximating the solution of operator equations that can be transformed into the form $Tx = x$, where X is an abstract space and $T : X \to X$ is an operator. The solution of an operator equation $Tx = x$ is known as *fixed point* of an operator T. The situation become more complex when an operator T on X does not assume values in a metric space X. If, we consider a nonself mapping $T : A \to B$ where A and B are disjoint nonempty subsets of (X, d), then an equation $Tx = x$ is not solvable and hence a need to obtain an optimal solution of such equation arises. This is achieved by minimizing the distance between Tx and x by solving the following (minimization) optimization problem:

$$\min_{x \in X} d(x, Tx). \tag{1}$$

The solution of the corresponding minimization problem (1) is known as an approximate solution of $Tx = x$ if T is a nonself operator. As, for each x in A, the distance between x and Tx cannot be less than the distance between A and B, so x satisfying

$$d(x, Tx) = d(A, B) \tag{2}$$

is the solution of minimization problem (1) and is called *best proximity point* of operator T, where

$$d\,(A, B) = \inf\{d\,(a, b) : a \in A,\ b \in B\},$$

is the measure of the distance between A and B.

In this manuscript, we assume that A and B are nonempty disjoint subsets of a metric space (X, d) and $T : A \to B$.

A point $x^* \in A$ is called *coincidence best proximity point* of a pair of mappings (g, T), if

$$d\,(gx^*, Tx^*) = d(A, B),$$

where $g : A \to A$.

Note that, *if $g = I_A$ (an identity mapping over A) then coincidence best proximity point becomes best proximity point.*

In 1965, Zadeh [18] introduced a fuzzy set. Later, Kramosil and Michálek [6] defined a fuzzy metric space combining the concept of a metric space with a fuzzy set. Afterwards, George and Veeramani [4] modified the definition provided in [6] and obtained a Hausdorff topology on fuzzy metric spaces. in 1989, Bakhtin [1] introduced the notion of $b-$metric space which can be viewed as a generalization of a metric space (see also, [3]). Sedghi and Shobe [14] combined the concepts of fuzzy set and $b-$metric to introduce $b-$fuzzy metric spaces, further Rakić *et al.* [9] provided a fixed point result in fuzzy b-metric space and Sedgi *et al.* [15] proved Suzuki type fixed point results in fuzzy metric spaces.

We need the following definitions, lemmas and results to prove our main results:

Definition 1.1. *[13] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous $t-$norm, if*

1) *$*$ is associative and commutative;*

2) *$*$ is continuous;*

3) $a * 1 = a$;

4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for any* $a, b, c, d \in [0, 1]$.

Some examples of continuous $t-$norms are $a*b = a\cdot b$ (usual product $t-$norm), $a*b = \min(a, b)$ (minimum $t-$norm) and $a * b = \max\{a + b - 1, 0\}$ (Lukasiewicz $t-$norm).

Definition 1.2. *[4] Let X be any nonempty set and $*$ a continuous $t-$norm. A fuzzy set M on $X \times X \times [0, \infty)$ is a fuzzy metric on X, if following conditions*

(FM1) $M(x, y, t) > 0$

(FM2) $M(x, y, t) = 1$ *if and only if* $x = y$

(FM3) $M(x, y, t) = M(y, x, t)$

(FM4) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$

(FM5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous,*

hold for all $x, y, z \in X$, *and* $s, t > 0$.

The set X equipped with a fuzzy metric M and a continuous $t-$norm $*$ is called a fuzzy metric space $(X, M, *)$. Here $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

In definition 1.2, if (FM4) is replaced with

$$M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s) \text{ for all } t, s > 0,$$

then M is called *non-Archimedean fuzzy metric* on X.

In definition 1.2, if (FM4) is replaced with

$$M(x, z, t + s) \geq M\left(x, y, \frac{t}{b}\right) * M\left(y, z, \frac{s}{b}\right),$$

then M is called $b-$*fuzzy metric* on X ([14]).

Remark 1.1. *The class of $b-$fuzzy metric spaces is effectively larger than the class of fuzzy metric spaces. Indeed, $b-$fuzzy metric space is a fuzzy metric space when $b = 1$ but converse does not hold in general.*

Notation 1.1. *[2] A class Ω consisting upon continuous and decreasing functions $\eta : (0, 1] \to [0, \infty)$, where $\eta(t) = 0$ if and only if $t = 1$ and $\eta(r * s) \leq \eta(r) + \eta(s)$ provided that $r * s > 0$ for $r, s \in (0, 1]$ and $*$ is any continuous $t-$norm.*

Definition 1.3. *[5] Let Ψ be a collection of continuous and nondecreasing functions $\psi : [0, 1] \to [0, 1]$, where $\psi(t) = 1$ if and only if $t = 1$, and $\psi(t) > t$, if $\lim_{n\to\infty} \psi^n(t) = 1$ for all $t \in (0, 1)$.*

Definition 1.4. *[5] Let $\beta : X \times X \times (0, \infty) \to (0, \infty)$ be any function. A mapping $g : X \to X$ is called a:*

1. *$\beta-$admissible, if*

$$\beta(x, y, t) \leq 1 \text{ implies that } \beta(gx, gy, t) \leq 1,$$

2. β_R- *admissible, if*
$$\beta(gx, gy, t) \le 1 \text{ implies that } \beta(x, y, t) \le 1,$$

for any $x, y \in X$, *and* $t > 0$.

Definition 1.5. *[8] Let* $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ *be any function. A mapping* $g : X \to X$ *is called an:*

1. $\alpha-$*admissible, if*
$$\alpha(x, y, t) \ge 1 \text{ implies } \alpha(gx, gy, t) \ge 1,$$
2. α_R-*admissible mapping, if*
$$\alpha(gx, gy, t) \ge 1 \text{ implies } a(x, y, t) \ge 1,$$

for any $x, y \in X$ *and* $t > 0$.

Definition 1.6. *[16] Let* $(X, M, *)$ *be a fuzzy metric space. For some* $x \in X$ *and* $t > 0$, *define*
$$\begin{aligned} A_0(t) &= \{x \in A : M(x, y, t) = M(A, B, t), \text{ for some } y \in B\}, \\ &\text{and} \\ B_0(t) &= \{y \in B : M(x, y, t) = M(A, B, t), \text{ for some } x \in A\}, \end{aligned}$$
where
$$M(A, B, t) = \sup\{M(a, b, t) \text{ for} a \in A \text{ and } b \in B\},$$
and
$$M(x, A, t) = \sup_{a \in A} M(x, a, t).$$

Definition 1.7. *([12]) Let* $\alpha : X \times X \times (0, \infty) \to [0, \infty)$. *A mapping* $T : A \to B$ *is said to be a fuzzy* $\alpha-$*proximal admissible, if*
$$\left.\begin{array}{l} \alpha(x, y, t) \ge 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{ implies that } \alpha(u, v, t) \ge 1,$$
for any $x, y, u, v \in A$ *and* $t > 0$.

Definition 1.8. *[11] A mapping* $g : A \to A$ *is said to be fuzzy expansive if for any* $x, y \in A$ *and* $t > 0$, *we have* $M(gx, gy, t) \le M(x, y, t)$.

Definition 1.9. *[10] Let* A *and* B *be two nonempty subsets of a fuzzy metric space* $(X, M, *)$. *A point* $x^* \in A$ *is said to be an optimal coincidence point of a pair of mappings* (g, T), *where* $g : A \to A$, *if*
$$M(gx^*, Tx^*, t) = M(A, B, t) \quad .$$

Definition 1.10. *[11] A set* B *in* X *is said to be fuzzy approximately compact with respect to* A *if for every sequence* $\{y_n\}$ *in* B *there exists some* $x \in A$, *such that if* $M(x, y_n, t) \to M(x, B, t)$, *then* $x \in A_0(t)$.

From now and onwards, X denotes the $b-$fuzzy metric space $(X, M, *, b)$ and (A, B) will denote a pair of nonempty subsets of a $b-$fuzzy metric space X (unless stated otherwise).

2 Main Results

Definition 2.1. *Let $T : A \to B,\ g : A \to A$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ be mappings. A pair of mappings (g, T) is said to be a fuzzy $(\alpha - \eta)-$generalized proximal contraction, if*

$$\left.\begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array}\right\} \text{implies that}$$

$$\alpha(x, y, t)\, \eta\left(M\left(gu, gv, t\right)\right) \leq k\eta\left(M\left(x, y, t\right)\right),$$

for all $x, y, u, v \in A,\ t > 0$, where $\eta \in \Omega$ and $k \in (0, 1)$.

Definition 2.2. *Let $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ be a mapping. A mapping $T : A \to B$ is said to be a fuzzy $(\alpha - \eta)-$proximal contraction, if*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{implies that}$$

$$\alpha(x, y, t)\, \eta\left(M\left(u, v, t\right)\right) \leq k\eta\left(M\left(x, y, t\right)\right),$$

for any $x, y, u, v \in A, t > 0$ where $\eta \in \Omega$ and $k \in (0, 1)$.

Remark 2.1. *Note that, if $g = I_A$ then fuzzy $(\alpha - \eta)-$generalized proximal contraction becomes fuzzy $(\alpha - \eta)-$proximal contraction.*

Definition 2.3. *Let $\beta : X \times X \times (0, \infty) \to (0, \infty)$. A mapping $T : A \to B$ is called a fuzzy $\beta-$proximal admissible, if*

$$\left.\begin{array}{c} \beta(x, y, t) \leq 1 \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{implies that } \beta(u, v, t) \leq 1,$$

for any $x, y, u, v \in A$ and $t > 0$.

Definition 2.4. *Let $T : A \to B,\ g : A \to A$ and $\beta : X \times X \times (0, \infty) \to (0, \infty)$. A pair of mappings (g, T) is said to be a fuzzy $(\beta - \psi)-$generalized proximal contraction, if*

$$\left.\begin{array}{l} M(gu, Tx, t) = M(A, B, t) \\ M(gv, Ty, t) = M(A, B, t) \end{array}\right\} \text{implies that}$$

$$\beta(x, y, t)\left(M\left(gu, gv, t\right)\right) \geq \psi(M\left(x, y, t\right)),$$

where $\psi \in \Psi$, $u, v, x, y \in A$, and $t > 0$.

Definition 2.5. *Let $\beta : X \times X \times (0, \infty) \to (0, \infty)$ be a mapping. A mapping $T : A \to B$ is said to be a fuzzy $(\beta - \psi)-$proximal contraction, if*

$$\left.\begin{array}{l} M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{array}\right\} \text{implies that}$$

$$\beta(x, y, t)\left(M\left(u, v, t\right)\right) \geq \psi(M\left(x, y, t\right)),$$

where $\psi \in \Psi$, $u, v, x, y \in A$, and $t > 0$.

Remark 2.2. *Note that, if $g = I_A$ then fuzzy $(\beta - \psi)-$generalized proximal contraction becomes fuzzy $(\beta - \psi)-$proximal contraction.*

In 2004, [7] introduced the notion of fuzzy $\psi-$contractive mappings and proved some interesting fixed point results in the setup of non-Archimedean fuzzy metric spaces. Later, [17] introduced fuzzy $\mathcal{H}-$contractive mapping as a generalization of fuzzy contractive mapping and obtained a fixed point result in $M-$complete fuzzy metric spaces.

In 2014, [5] introduced the notions of $(\alpha-\phi)-$fuzzy contractive mapping and $(\beta-\psi)-$fuzzy contractive mappings. Further, they studied the sufficient conditions for the existence and uniqueness of a fixed point of such mappings.

Motivated by the work of [5], in this present paper, we introduced the concepts of fuzzy $(\alpha-\eta)-$proximal contraction, fuzzy $(\alpha-\eta)-$generalized proximal contraction, fuzzy $(\beta-\psi)-$proximal contraction and fuzzy $(\beta-\psi)-$generalized proximal contraction and obtain some best proximity point and coincidence point results for such mappings in $b-$fuzzy metric space.

We need the following lemmas and results for main results.

Lemma 2.1. *Let $A_0(t)$ and $B_0(t)$ be nonempty subsets of a $b-$fuzzy metric space X, $T: A\to B$ a fuzzy $\alpha-$proximal admissible mapping and $g: A\to A$ a fuzzy α_R-admissible mapping with $T(A_0(t))\subseteq B_0(t)$, $A_0(t)\subseteq g(A_0(t))$. If $M(gx_1,Tx_0,t)=M(A,B,t)$ and $\alpha(x_1,x_0,t)\geq 1$, for any $x_0,x_1\in A_0(t)$, then there exists a sequence $\{x_n\}\subset A_0(t)$ such that*

$$M(gx_{n+1},Tx_n,t)=M(A,B,t)\ \text{and}\ \alpha(x_{n+1},x_n,t)\geq 1, \tag{3}$$

for $n\in\mathbb{N}$ with $x_n,x_{n+1}\in A_0(t)$.

Proof. As $x_0,x_1\in A_0(t)\subseteq g(A_0(t))$ such that $M(gx_1,Tx_0,t)=M(A,B,t)$ with $\alpha(x_1,x_0,t)\geq 1$. For $Tx_1\in T(A_0(t))\subseteq B_0(t)$, there exists $x_2\in A_0(t)$ such that $M(gx_2,Tx_1,t)=M(A,B,t)$. Since T is fuzzy $\alpha-$proximal admissible, therefore

$$\left.\begin{array}{l}\alpha(x_1,x_0,t)\geq 1,\\ M(gx_1,Tx_0,t)=M(A,B,t)\\ M(gx_2,Tx_1,t)=M(A,B,t)\end{array}\right\}\text{ implies that }\alpha(gx_2,gx_1,t)\geq 1.$$

As g is an α_R-admissible, so $\alpha(x_2,x_1,t)\geq 1$. Similarly,

$$\left.\begin{array}{l}\alpha(x_2,x_1,t)\geq 1\\ M(gx_2,Tx_1,t)=M(A,B,t)\\ M(gx_3,Tx_2,t)=M(A,B,t)\end{array}\right\}\text{implies that }\alpha(gx_3,gx_2,t)\geq 1,$$

so $\alpha(x_3,x_2,t)\geq 1$. Continuing this way, we obtain a sequence $\{x_n\}\subset A_0(t)$ which satisfies

$$M(gx_{n+1},Tx_n,t)=M(A,B,t)\ \text{and}\ \alpha(x_{n+1},x_n,t)\geq 1,$$

for all $n\in\mathbb{N}$ with $x_n,x_{n+1}\in A_0(t)$. □

Definition 2.6. *A sequence $\{x_n\}\subset A_0(t)$ satisfying the condition (3) is called an $\alpha-$proximal sequence starting with $x_0\in A_0(t)$.*

Lemma 2.2. *Let $A_0(t)$ and $B_0(t)$ be nonempty subsets of a $b-$fuzzy metric space X, $T: A\to B$ a fuzzy $\beta-$proximal admissible mapping and $g: A\to A$ a fuzzy β_R-admissible mapping with $T(A_0(t))\subseteq B_0(t)$, $A_0(t)\subseteq g(A_0(t))$. If $M(gx_1,Tx_0,t)=M(A,B,t)$ and $\beta(x_1,x_0,t)\leq 1$, for any $x_0,x_1\in A_0(t)$. Then, there exists a sequence $\{x_n\}\subset A_0(t)$ such that*

$$M(gx_{n+1},Tx_n,t)=M(A,B,t)\ \text{and}\ \beta(x_{n+1},x_n,t)\leq 1, \tag{4}$$

for $n\in\mathbb{N}$ with $x_n,x_{n+1}\in A_0(t)$.

Proof. As $x_0, x_1 \in A_0(t) \subseteq g(A_0(t))$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ with $\beta(x_1, x_0, t) \leq 1$. For $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, there exists $x_2 \in A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Since T is fuzzy $\beta-$proximal admissible, therefore

$$\left.\begin{array}{l}\beta(x_1, x_0, t) \leq 1, \\ M(gx_1, Tx_0, t) = M(A, B, t) \\ M(gx_2, Tx_1, t) = M(A, B, t)\end{array}\right\} \text{implies that } \beta(gx_2, gx_1, t) \leq 1.$$

Since g is β_R-admissible, so $\beta(x_2, x_1, t) \leq 1$. Similarly,

$$\left.\begin{array}{l}\beta(x_2, x_1, t) \leq 1 \\ M(gx_2, Tx_1, t) = M(A, B, t) \\ M(gx_3, Tx_2, t) = M(A, B, t)\end{array}\right\} \text{implies that } \beta(gx_3, gx_2, t) \leq 1,$$

which further implies that $\beta(x_3, x_2, t) \leq 1$. Continuing this way, we obtain a sequence $\{x_n\} \subset A_0(t)$ which satisfies

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t) \text{ and } \beta(x_{n+1}, x_n, t) \leq 1,$$

for $n \in \mathbb{N}$ with $x_n, x_{n+1} \in A_0(t)$. □

Definition 2.7. *A sequence $\{x_n\} \subset A_0(t)$ satisfying the condition (4) is called $\beta-$proximal sequence starting with $x_0 \in A_0(t)$.*

3 Optimal coincidence point results for fuzzy $(\alpha - \eta)-$generalized proximal contraction

In this section, we prove an optimal coincidence best proximity point result for a pair of mappings (g, T) satisfying fuzzy $(\alpha - \eta)-$generalized proximal contractive condition in the setting of a complete $b-$fuzzy metric space X.

Theorem 3.1. *Let $T : A \to B$ and $g : A \to A$ be a fuzzy expansive mapping, where A is a nonempty closed subset of a complete $b-$fuzzy metric space X and B is a fuzzy approximately compact with respect to A with $A_0(t) \subseteq g(A_0(t))$ and $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If there exist $x_0, x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_1, x_0, t) \geq 1$. The pair of mappings (g, T) is a fuzzy $(\alpha - \eta)-$generalized proximal contraction. Then, mappings g and T have a unique optimal coincidence point x^* in $A_0(t)$.*

Proof. As $A_0(t)$ is nonempty and $T(A_0(t)) \subseteq B_0(t)$ with $A_0(t) \subseteq g(A_0(t))$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_1, x_0, t) \geq 1$. Then by lemma (2.1), there exists a sequence $\{x_n\} \subset A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t) \text{ and } \alpha(x_{n+1}, x_n, t) \geq 1, \tag{5}$$

with $x_n, x_{n+1} \in A_0(t)$ and for each $n \in \mathbb{N} \cup \{0\}$. Since (g, T) is fuzzy $(\alpha - \eta)-$ generalized proximal contraction, we have

$$\alpha(x_{n-1}, x_n, t)\, \eta(M(gx_n, gx_{n+1}, t)) \leq k\eta(M(x_{n-1}, x_n, t)), \text{ for all } n \geq 0.$$

Since g is fuzzy expansive and η is decreasing mapping, the above inequality can be written as

$$\begin{aligned}\eta(M(x_n, x_{n+1}, t)) &\leq \eta(M(gx_n, gx_{n+1}, t)) \\ &\leq \alpha(x_{n-1}, x_n, t)\, \eta(M(gx_n, gx_{n+1}, t)) \\ &\leq k\eta(M(x_{n-1}, x_n, t)),\end{aligned}$$

which implies that

$$\begin{aligned}\eta\left(M\left(x_n,x_{n+1},t\right)\right) &\leq k\eta\left(M\left(x_{n-1},x_n,t\right)\right)\\ &< \eta\left(M\left(x_{n-1},x_n,t\right)\right), \text{ for some } k\in(0,1).\end{aligned}$$

Further, we have

$$\begin{aligned}\eta\left(M\left(x_n,x_{n+1},t\right)\right) &\leq \alpha\left(x_{n-1},x_n,t\right)\eta\left(M\left(gx_n,gx_{n+1},t\right)\right)\\ &\leq k\eta\left(M\left(x_{n-1},x_n,t\right)\right)\\ &\leq k\left[\alpha\left(x_{n-2},x_{n-1},t\right)\eta\left(M\left(gx_{n-1},gx_n,t\right)\right)\right]\\ &\leq k^2\eta\left(M\left(x_{n-2},x_{n-1},t\right)\right)\\ &\leq \vdots\\ &\leq k^n\eta\left(M\left(x_0,x_1,t\right)\right), \text{ for all } t>0.\end{aligned}$$

Since $k\in(0,1)$ and η is strictly decreasing, we have

$$\eta\left(M\left(x_n,x_{n+1},t\right)\right)<\eta\left(M\left(x_0,x_1,t\right)\right), \text{ for all} t>0,$$

and

$$M\left(x_n,x_{n+1},t\right)\geq M\left(x_0,x_1,t\right), \text{for all } t>0,\ n\in\mathbb{N}. \tag{6}$$

Let $m,n\in\mathbb{N}$ with $m<n$. Suppose that $\{a_i\}$ is a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i=1$. For any real number $b\geq 1$, we have

$$\begin{aligned}M\left(x_n,x_m,t\right) &\geq M\left(x_n,x_n,\frac{t}{b}-\sum_{i=n}^{m-1}\frac{a_it}{b}\right)*M\left(x_n,x_m,\sum_{i=n}^{m-1}\frac{a_it}{b}\right)\\ &\geq M\left(x_n,x_{n+1},\frac{a_nt}{b^2}\right)*M\left(x_{n+1},x_{n+2},\frac{a_{n+1}t}{b^2}\right)*\cdots*\\ &\quad M\left(x_{m-1},x_m,\frac{a_{m-1}t}{b^2}\right).\end{aligned}$$

Thus

$$\begin{aligned}\eta\left(M\left(x_n,x_m,t\right)\right) &\leq \eta\left(\prod_{i=n}^{m-1}M\left(x_i,x_{i+1},\frac{a_it}{b^2}\right)\right)\\ &\leq \sum_{i=n}^{m-1}\eta\left(M\left(x_i,x_{i+1},\frac{a_it}{b^2}\right)\right)\\ &\leq \sum_{i=n}^{m-1}k^i\eta\left(M(x_0,x_1,\frac{a_it}{b^2})\right), \text{ where } t>0.\end{aligned}$$

Consider

$$q=\max\{\frac{a_i}{b^2}:n\leq i\leq m-1 \text{ and } b\geq 1\}.$$

Then

$$\eta\left(M\left(x_n,x_m,t\right)\right)\leq\sum_{i=n}^{m-1}k^i\eta\left(M(x_0,x_1,qt)\right)<\epsilon, \text{ for all } t>0, \text{ and } \epsilon>0.$$

Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a closed subset A of a complete $b-$fuzzy metric space X. Hence there exists some $x^*\in A_0(t)\subset A$ such that

$$\lim_{n\to\infty}M\left(x_n,x^*,t\right)=1, \text{ for all } t>0.$$

Since

$$\begin{aligned} M\left(gx_{n+1}, B, t\right) &\geq M\left(gx_{n+1}, Tx_n, t\right) \\ &= M\left(A, B, t\right) \\ &\geq M\left(gx_{n+1}, B, t\right). \end{aligned}$$

As g is continuous and the sequence $\{x_n\}$ converges to x^*, hence the sequence $\{gx_n\}$ converges to $g(x^*)$,

$$M\left(gx^*, Tx^*, t\right) \to M(gx^*, B, t).$$

By taking $\{y_n\} = \{Tx^*, Tx^*, Tx^* \ldots\}$ (say) for all $n \in \mathbb{N} \cup \{0\}$. Since $\{Tx_n\} \subseteq B$ and B is a fuzzy approximately compact with respect to A, therefore $M\left(gx^*, y_n, t\right) = M\left(gx^*, B, t\right)$, and hence $gx^* \in A_0\left(t\right)$. Now $A_0 \subseteq g\left(A_0\right)$ gives some $u \in A_0\left(t\right)$ such that

$$M\left(gu, Tx^*, t\right) = M\left(A, B, t\right) = M\left(gx_{n+1}, Tx_n, t\right), \text{for all } n \in \mathbb{N}.$$

As (g, T) is fuzzy $(\alpha - \eta)-$generalized proximal contraction and g is fuzzy expansive mapping, we have

$$\eta\left(M\left(u, x_{n+1}, t\right)\right) \leq \alpha\left(x^*, x_n, t\right) \eta\left(M\left(gu, gx_{n+1}, t\right)\right) \leq k\eta\left(M\left(x^*, x_n, t\right)\right).$$

On taking limit as $n \to \infty$ on both sides of the above inequality, we obtain

$$\eta\left(M\left(u, x^*, t\right)\right) \leq 0,$$

and hence $M\left(u, x^*, t\right) = 1$ which implies that $u = x^*$. Thus

$$M\left(gx^*, Tx^*, t\right) = M\left(gu, Tx^*, t\right) = M\left(A, B, t\right),$$

implies that x^* is the optimal coincidence point of the pair (g, T).

Uniqueness: Suppose that there exists another optimal coincidence point $y^* \neq x^*$ of the pair (g, T) in $A_0\left(t\right)$. Then, we have

$$M\left(gx^*, Tx^*, t\right) = M\left(A, B, t\right) = M\left(gy^*, Ty^*, t\right) \text{ and }, \alpha\left(x^*, y^*, t\right) \geq 1.$$

Since (g, T) is fuzzy $(\alpha - \eta)-$generalized proximal contraction and g is fuzzy expansive mapping, we have

$$\begin{aligned} \eta\left(M\left(x^*, y^*, t\right)\right) \leq \alpha\left(x^*, y^*, t\right) \eta\left(M\left(gx^*, gy^*, t\right)\right) &\leq k\eta\left(M\left(x^*, y^*, t\right)\right) \\ &< \eta\left(M\left(x^*, y^*, t\right)\right), \end{aligned}$$

a contradiction. Hence the optimal coincidence point of the pair (g, T) is unique. □

Example 3.1. *Let* $X = [0, 1] \times \mathbb{R}$, $A = \{(x, 1) : x \in \mathbb{R}\}$ *and* $B = \{(x, -1) : x \in \mathbb{R}\}$. *Define* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ *by*

$$d\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \left(|x_1 - x_2| + |y_1 - y_2|\right)^2.$$

Note that, $(X, M, \wedge, b)$ *is a complete* $b-$*fuzzy metric space, where standard* $b-$*fuzzy metric* $M(x, y, t)$ *induced by* d*, is given by*

$$M\left(x, y, t\right) = \frac{t}{t + d\left(x, y\right)}.$$

It is straight forward to check that

$$M\left(A, B, t\right) = \frac{t}{t + 4}.$$

Note that, $A_0(t) = A$ *and* $B_0(t) = B$. *Define* $T : A \to B$ *and* $g : A \to A$ *as*

$$T(x,1) = \left(\frac{x}{3}, -1\right) \text{ and } g(x,1) = (3x,1).$$

Clearly g *is a fuzzy expansive mapping,* $T(A_0(t)) \subseteq B_0(t)$ *and* $A_0(t) = g(A_0(t))$. *If we take* $u = (x_1, 1)$, $v = (x_2, 1) \in A$, *then there exist* $x = (x_3, 1)$ *and* $y = (x_4, 1) \in A$, *such that*

$$M(gu, Tx, t) = M(A, B, t) = M(gv, Ty, t),$$

holds for $x_1 = \frac{x_3}{9}$ *and* $x_2 = \frac{x_4}{9}$. *Define* $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ *by*

$$\alpha(x,y,t) = \begin{cases} 1, & \text{if } x,y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

If we take $\eta(t) = \frac{1}{t} - 1$, *then*

$$\alpha(x,y,t)\,\eta(M(gu,gv,t)) \le k\eta(M(x,y,t))$$

is satisfied for $k = \frac{1}{9}$. *Thus all the conditions of the Theorem (3.1) are satisfied. Moreover* $(0,1)$ *is the unique coincidence point of* (g,T) *in* $A_0(t)$.

Corollary 3.1. *Let* $T : A \to B$ *and* $g : A \to A$ *be a fuzzy isometric mapping, where* A *is a nonempty closed subset of a complete* $b-$*fuzzy metric space* X *and* B *is a fuzzy approximately compact with respect to* A *with* $\phi \neq A_0(t) \subseteq g(A_0(t))$ *and* $T(A_0(t)) \subseteq B_0(t)$ *for each* $t > 0$. *If there exist* $x_0, x_1 \in A_0(t)$, *such that* $M(gx_1, Tx_0, t) = M(A, B, t)$ *and* $\alpha(x_1, x_0, t) \ge 1$. *The pair of mappings* (g,T) *is a fuzzy* $(\alpha - \eta)-$*generalized proximal contraction. Then, mappings* g *and* T *have a unique optimal coincidence point* x^* *in* $A_0(t)$.

Proof. Since mapping g is a fuzzy isometry, that is for any $x, y \in A$ and $t > 0$, $M(gx, gy, t) = M(x, y, t)$. Thus, g is fuzzy expansive on A. The proof of the result follows on the same lines as Theorem (3.1). □

Corollary 3.2. *Let* $T : A \to B$ *be a fuzzy* $(\alpha - \eta)-$*proximal contraction, where* A *is a nonempty closed subset of a complete* $b-$*fuzzy metric space* X *and* B *is a fuzzy approximately compact with respect to* A *with* $\phi \neq A_0(t)$ *and* $T(A_0(t)) \subseteq B_0(t)$ *for each* $t > 0$. *If there exist* $x_0, x_1 \in A_0(t)$, *such that* $M(x_1, Tx_0, t) = M(A, B, t)$ *and* $\alpha(x_1, x_0, t) \ge 1$. *Then, there exists a unique best proximity point of the mapping* T.

Proof. If we take $g = I_A$ (an identity mapping on A) then the remaining proof of this corollary follows on the same lines as Theorem (3.1). □

4 Optimal coincidence point results for fuzzy $(\beta - \psi)-$generalized proximal contraction

In this section, we prove an optimal coincidence best proximity point result for a pair of mappings (g,T) which satisfies fuzzy $(\beta - \psi)-$generalized proximal contractive condition in the setting of a complete $b-$fuzzy metric space X.

Theorem 4.1. *Let $T : A \to B$ and $g : A \to A$ be a fuzzy expansive mapping, where A is a nonempty closed subset of a complete $b-$fuzzy metric space X and B is a fuzzy approximately compact with respect to A with $A_0(t) \subseteq g(A_0(t))$ and $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If there exist $x_0, x_1 \in A_0(t)$, such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_1, x_0, t) \geq 1$ provided that pair of mappings (g, T) is a fuzzy $(\beta - \psi)-$generalized proximal contraction. Then, mappings g and T have a unique optimal coincidence point x^* in $A_0(t)$.*

Proof. As $A_0(t) \neq \phi$, $T(A_0(t)) \subseteq B_0(t)$ with $A_0(t) \subseteq g(A_0(t))$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\beta(x_1, x_0, t) \leq 1$. Then by lemma (2.2), there exist a sequence $\{x_n\} \subset A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t) \text{ and } \beta(x_{n+1}, x_n, t) \leq 1, \tag{7}$$

with $x_n, x_{n+1} \in A_0(t)$ for each $n \in \mathbb{N} \cup \{0\}$. Since the pair (g, T) is fuzzy $(\beta - \psi)-$generalized proximal contraction, we have

$$\beta(x_n, x_{n-1}, t) M(gx_n, gx_{n-1}, t) \geq \psi(M(x_n, x_{n-1}, t)), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

As g is fuzzy expansive, ψ is nondecreasing and continuous on $[0, 1]$,

$$M(x_{n+1}, x_n, t) \geq M(gx_{n+1}, gx_n, t) \geq \psi(M(x_n, x_{n-1}, t)),$$

implies that,

$$M(x_{n+1}, x_n, t) \geq \psi(M(x_n, x_{n-1}, t)) > M(x_n, x_{n-1}, t).$$

Note that

$$\begin{aligned} M(x_{n+1}, x_n, t) &\geq \beta(x_n, x_{n-1}, t) M(gx_{n+1}, gx_n, t) \\ &\geq \psi(M(x_n, x_{n-1}, t)) \\ &\geq \psi[\beta(x_{n-1}, x_{n-2}, t) M(gx_n, gx_{n-1}, t)] \\ &\geq \psi^2(M(x_{n-1}, x_{n-2}, t)) \\ &\geq \vdots \\ &\geq \psi^n(M(x_1, x_0, t)), \text{ for all } n \in \mathbb{N}, t > 0. \end{aligned}$$

Hence

$$M(x_{n+1}, x_n, t) \geq \psi^n(M(x_1, x_0, t)).$$

Since $\lim_{n \to +\infty} \psi^{n+1}(r) = 1$ for all $r \in (0, 1)$, we obtain that

$$\lim_{n \to +\infty} M(x_{n+1}, x_n, t) = 1, \text{ for all } t > 0.$$

Now, we have to show that $\{x_n\}$ is a Cauchy sequence. On the contrary suppose that the sequence $\{x_n\}$ is not a Cauchy, then there exists $\epsilon \in (0, 1)$, $t_0 > 0$ for each $k, k_0 \in \mathbb{N}$ with $k \geq k_0$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k)$ such that

$$M\left(x_{m(k)}, x_{n(k)}, t\right) \leq 1 - \epsilon.$$

Clearly, $\beta\left(x_{n(k)}, x_{n(k)-1}, t\right) \leq 1$. Assume that $m(k)$ is the least such integer exceeding $n(k)$, then for each k, we have

$$M\left(x_{m(k)}, x_{n(k)}, t\right) \leq 1 - \epsilon, \text{ and } M\left(x_{m(k)-1}, x_{n(k)}, t\right) > 1 - \epsilon.$$

Then, for each positive integer $k \geq k_0$, we have

$$\begin{aligned}
1-\epsilon &\geq M\left(x_{m(k)}, x_{n(k)}, t\right) \\
&\geq M\left(x_{m(k)}, x_{m(k)-1}, \frac{t}{2b}\right) * M\left(x_{m(k)-1}, x_{n(k)}, \frac{t}{2b}\right) \\
&\geq M\left(x_{m(k)}, x_{m(k)-1}, \frac{t}{2b}\right) * (1-\epsilon),
\end{aligned}$$

On taking limit as $k \to \infty$ on both sides of the above inequality, we have

$$\lim_{n\to+\infty} M\left(x_{m(k)}, x_{n(k)}, t\right) = 1-\epsilon.$$

Now,

$$\begin{aligned}
M\left(x_{m(k)}, x_{n(k)}, t\right) &\geq M\left(x_{m(k)}, x_{m(k)+1}, \frac{t}{3b}\right) * M\left(x_{m(k)+1}, x_{n(k)+1}, \frac{t}{3b}\right) * \\
&\quad M\left(x_{n(k)+1}, x_{n(k)}, \frac{t}{3b}\right) \\
&\geq M\left(x_{m(k)}, x_{m(k)+1}, \frac{t}{3b}\right) * M\left(x_{n(k)+1}, x_{n(k)}, \frac{t}{3b}\right) * \\
&\quad \beta\left(x_{n(k)+1}, x_{n(k)}, \frac{t}{3b}\right) M\left(gx_{m(k)+1}, gx_{n(k)+1}, \frac{t}{3b}\right) \\
&\geq M\left(x_{m(k)}, x_{m(k)+1}, \frac{t}{3b}\right) * \psi\left(M\left(x_{n(k)+1}, x_{n(k)}, \frac{t}{3b}\right)\right) * \\
&\quad M\left(x_{n(k)+1}, x_{n(k)}, \frac{t}{3b}\right).
\end{aligned}$$

Taking limit as $k \to \infty$ on both side of the above inequality, we have

$$1-\epsilon \geq 1 * \psi(1-\epsilon) * 1 = \psi(1-\epsilon) > 1-\epsilon,$$

a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in a closed subset A of a complete $b-$fuzzy metric space X. There exists some $x^* \in A$ such that

$$\lim_{n\to+\infty} M(x_n, x^*, t) = 1, \text{ for all } t > 0.$$

Also, we have

$$M(gx_{n+1}, B, t) \geq M(gx_{n+1}, Tx_n, t) = M(A, B, t) \geq M(gx_{n+1}, B, t).$$

As g is continuous and the sequence $\{x_n\}$ converges to x^*, the sequence $\{gx_n\}$ converges to $g(x^*)$, we have

$$M(gx^*, Tx^*, t) \to M(gx^*, B, t),$$

taking $\{y_n\} = \{Tx^*, Tx^*, Tx^*, \cdots\}$ (say) for all $n \in \mathbb{N} \cup \{0\}$. Since $\{Tx_n\} \subseteq B$, and B is a fuzzy approximately compact with respect to A, $\{Tx_n\}$ has a subsequence which converges to some y in B, therefore $M(gx^*, y_n, t) = M(A, B, t)$, and hence $gx^* \in A_0(t)$. Now $A_0 \subseteq g(A_0)$ gives that

$$M(gu, Tx^*, t) = M(A, B, t) = M(gx_{n+1}, Tx_n, t), \text{for all } n \in \mathbb{N},$$

for some $u \in A_0(t)$. Since (g, T) is fuzzy $(\beta-\psi)-$ generalized proximal contraction and g is fuzzy expansive, we have

$$M(u, x_{n+1}, t) \geq \beta(x^*, x_n, t) M(gu, gx_{n+1}, t) \geq \psi(M(x^*, x_n, t)).$$

Taking limit as $n \to \infty$ on both sides of the above inequality, we have

$$M(u, x^*, t) \geq 1,$$

and hence $M(u, x^*, t) = 1$ which implies that $u = x^*$. Now

$$M(gx^*, Tx^*, t) = M(gu, Tx^*, t) = M(A, B, t),$$

shows that x^* is the optimal coincidence point of the pair of mappings (g, T).

Uniqueness: If there exist another optimal coincidence point $y^* \neq x^*$ of the pair (g, T) in $A_0(t)$, then we have

$$M(gx^*, Tx^*, t) = M(A, B, t) = M(gy^*, Ty^*, t) \text{ and } , \beta(x^*, y^*, t) \leq 1.$$

Since (g, T) is a fuzzy $(\beta - \psi)-$generalized proximal contraction and g is fuzzy expansive, so

$$M(x^*, y^*, t) \geq \beta(x^*, y^*, t) M(gx^*, gy^*, t) \geq \psi(M(x^*, y^*, t)) > (M(x^*, y^*, t)),$$

a contraction. Hence optimal coincidence point of the pair (g, T) is unique. □

Example 4.1. *Let* $X = [0, 1] \times \mathbb{R}$, $A = \{(x, 1) : x \in \mathbb{R}\}$ *and* $B = \{(x, -1) : x \in \mathbb{R}\}$. *Define* $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ *by*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}.$$

Note that, $(X, M, \wedge, b)$ *is a complete* $b-$*fuzzy metric space, where standard* $b-$*fuzzy metric* $M(x, y, t)$ *induced by* d*, is given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

It is straight forward to check that

$$M(A, B, t) = \frac{t}{t + 2}.$$

Clearly $A_0(t) = A$, $B_0(t) = B$. *Define mappings* $T : A \to B$ *and* $g : A \to A$ *by*

$$T(x, 1) = \left(\frac{x}{3}, -1\right) \text{ and } g(x, 1) = (3x, 1).$$

Clearly g *is fuzzy expansive,* $T(A_0(t)) \subseteq B_0(t)$ *and* $A_0(t) \subseteq g(A_0(t))$. *If we take* $u = (x_1, 1)$, $v = (x_2, 1) \in A$, *then there exist* $x = (x_3, 1)$ *and* $y = (x_4, 1) \in A$ *such that*

$$M(gu, Tx, t) = M(A, B, t) = M(gv, Ty, t),$$

holds for $x_1 = \frac{x_3}{9}$ *and* $x_2 = \frac{x_4}{9}$. *Define*

$$\beta(x, y, t) = \begin{cases} 1, & \text{if } x,y \in [0,1], \\ 2, & \text{otherwise.} \end{cases}$$

If, we take $\psi(t) = \sqrt{t}$, *then*

$$\beta(x, y, t) M(gu, gv, t) \geq \psi(M(x, y, t))$$

holds. Thus all the conditions of Theorem (4.1) are satisfied. Moreover $(0, 1)$ *is the unique coincidence point of* (g, T) *in* $A_0(t)$.

Corollary 4.1. *Let $T : A \to B$ and $g : A \to A$ be a fuzzy isometric mapping, where A is a nonempty closed subset of a complete $b-$fuzzy metric space X and B is a fuzzy approximately compact with respect to A with $\phi \neq A_0(t) \subseteq g(A_0(t))$ and $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If there exist $x_0, x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$ and $\alpha(x_1, x_0, t) \geq 1$. Then, mappings g and T have a unique optimal coincidence point x^* in $A_0(t)$ provided that (g, T) is fuzzy $(\beta - \psi)-$generalized proximal contraction.*

Proof. Since mapping g is a fuzzy isometry, that is for any $x, y \in A$ and $t > 0$, $M(gx, gy, t) = M(x, y, t)$. Thus, g is fuzzy expansive on A. The proof of the corollary follows on the same lines as given in Theorem (4.1). □

Corollary 4.2. *Let $T : A \to B$ be a fuzzy $(\beta - \psi)-$proximal contraction, where A is a nonempty closed subset of a complete $b-$fuzzy metric space X and B is a fuzzy approximately compact with respect to A with $\phi \neq A_0(t)$ and $T(A_0(t)) \subseteq B_0(t)$ for each $t > 0$. If there exist $x_0, x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$ and $\beta(x_1, x_0, t) \leq 1$. Then, there exists a unique best proximity point of the mapping T.*

5 Conclusion

In this article, we defined fuzzy $(\alpha-\eta)$ and fuzzy $(\beta-\psi)-$generalized proximal contractions in complete $b-$fuzzy metric spaces and proved the existence of coincidence and best proximity point of such mappings. Some examples are provided to show that the results presented herein generalize and extend comparable results to nonself mappings. It is worth mentioning that the results in [2, 5] are extended if we restrict ourselves to self mappings.

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